Translation-finite sets, and weakly compact derivations from $\ell^1(\mathbb{Z}_+)$ to its dual

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12th October 2009†

Abstract

We characterize those derivations from the convolution algebra $\ell^1(\mathbb{Z}_+)$ to its dual which are weakly compact, providing explicit examples which are not compact. The characterization is combinatorial, in terms of "translation-finite" subsets of $\mathbb{Z}_+$, and we investigate how this notion relates to other notions of "smallness" for infinite subsets of $\mathbb{Z}_+$. In particular, we prove that a set of strictly positive Banach density cannot be translation-finite; the proof has a Ramsey-theoretic flavour.

1 Introduction

The problem of determining the weakly compact or compact homomorphisms between various Banach algebras has been much studied; the study of weakly compact or compact derivations, less so. In certain cases, the geometrical properties of the underlying Banach space play an important role. For instance, by a result of Morris [9], every bounded derivation from the disc algebra $A(\mathbb{D})$ to its dual is automatically weakly compact. (It had already been shown in [2] that every bounded operator from $A(\mathbb{D})$ to $A(\mathbb{D})^*$ is automatically 2-summing, hence weakly compact; but the proof is significantly harder than that of the weaker result in [9].)

In this article, we investigate the weak compactness or otherwise of derivations from the convolution algebra $\ell^1(\mathbb{Z}_+)$ to its dual. Unlike the case of $A(\mathbb{D})$, the space of derivations is easily parametrized: every bounded derivation from $\ell^1(\mathbb{Z}_+)$ to its dual is of the form

$$D_\psi(\delta_0) = 0 \quad \text{and} \quad D_\psi(\delta_j)(\delta_k) = \frac{j}{j+k}\psi_{j+k} \qquad (j \in \mathbb{N}, k \in \mathbb{Z}_+) \tag{1.1}$$

for some $\psi \in \ell^\infty(\mathbb{N})$. It was shown in the second author's thesis [7] that D_ψ is compact if and only if $\psi \in c_0$, and that there exist ψ for which D_ψ is not weakly compact.

The primary purpose of the present note is to characterize those ψ for which D_ψ is weakly compact (see Theorem 2.6 below). In particular, we show that there exist a plethora of ψ for which D_ψ is weakly compact but not compact. Our criterion is combinatorial and uses the notion, apparently due to Ruppert, of translation-finite subsets of

*The 2nd author is supported by post-doctoral grant SFRH/BPD/40762/2007 from FCT (Portugal).

†MSC2000: Primary 43A20, 43A46, 47B07

a semigroup. A secondary purpose is to construct various examples of translation-finite and non-translation-finite subsets of $\mathbb{Z}_+$, to clarify the connections or absence thereof with other combinatorial notions of "smallness".

An example

We first resolve a question from [7], by giving a very simple example of a D_ψ that is non-compact but is weakly compact.

Proposition 1.1 *Let ψ be the indicator function of $\{2^n : n \in \mathbb{N}\} \subset \mathbb{N}$. Then D_ψ is non-compact, and the range of D_ψ is contained in $\ell^1(\mathbb{Z}_+)$.*

In particular, since $\ell^1(\mathbb{Z}_+) \subset \ell^p(\mathbb{Z}_+)$ for every $1 < p < \infty$, D_ψ factors through a reflexive Banach space and is therefore weakly compact.

Proof. Since $\psi \notin c_0$, we know by [7, Theorem 5.7.3] that D_ψ is non-compact.

We have $D_\psi(\delta_0) = 0$. For each $j \in \mathbb{N}$, let $N_j = \min(n \in \mathbb{N} \,:\, 2^n \geq j)$; then

$$\|D_\psi(\delta_j)\|_1 = \sum_{k\geq 0} \left| \frac{j}{(j+k)} \psi_{j+k} \right| = \sum_{n \geq N_j} \frac{j}{2^n} = \frac{j}{2^{N_j - 1}} \leq 2\,.$$

By linearity and continuity we deduce that $\|D_\psi(a)\|_1 \leq 2\|a\|_1$ for all $a \in \ell^1(\mathbb{Z}_+)$. The last assertion now follows, by standard results on weak compactness of operators. $\Box$

Remark 1.2 Since D_ψ factors through the *inclusion* map $\ell^1(\mathbb{Z}_+) \to c_0(\mathbb{Z}_+)$, which is known to be 1-summing, it too is 1-summing.

This last remark raises the natural question: is *every* weakly compact derivation from $\ell^1(\mathbb{Z}_+)$ to its dual automatically p-summing for some $p < \infty$? The answer, unsurprisingly, is negative: we have deferred the relevant counterexample to an appendix.

2 Characterizing weakly compact derivations

We need only the basic results on weak compactness in Banach spaces, as can be found in standard references such as [8].

Recall that if X is a closed subspace of a Banach space Y, and $K \subseteq X$, then K is weakly compact as a subset of X if and only if it is weakly compact as a subset of Y. Since (by Equation (1.1)) our derivations D_ψ take values in $c_0(\mathbb{Z}_+)$, we may therefore work with the weak topology of $c_0(\mathbb{Z}_+)$ rather than that of $\ell^\infty(\mathbb{Z}_+)$.

Moreover, we can reduce the verification of weak compactness to that of sequential pointwise compactness. This is done through some simple lemmas, which we give below.

Lemma 2.1 *Let (y_i) be a bounded net in $c_0(\mathbb{Z}_+)$, and let $y \in c_0(\mathbb{Z}_+)$. Then (y_i) converges weakly to y if and only if it converges pointwise to y.*

The proof is straightforward and we omit the details.

Lemma 2.2 *Let $T : \ell^1(\mathbb{Z}_+) \to c_0(\mathbb{Z}_+)$ be a bounded linear map. Then the following are equivalent:*

(i) *T is weakly compact;*

(ii) *every subsequence of $(T(\delta_n))_{n\in\mathbb{N}}$ has a further subsequence which converges pointwise to some $y \in c_0(\mathbb{Z}_+)$.*

Proof. Let B denote the closed unit ball of $\ell^1(\mathbb{Z}_+)$, let $\mathsf{E} = \{T(\delta_n) : n \in \mathbb{N}\}$, and let τ denote the topology of pointwise convergence in c_0. Note that the restriction of τ to bounded subsets of c_0 is a *metrizable* topology.

If (i) holds, then by (the trivial half of) Lemma 2.1, the bounded set $T(\mathsf{B})$ is τ-precompact, and hence sequentially τ-precompact. Thus (ii) holds.

Conversely, suppose that (ii) holds, i.e. that E is sequentially τ-precompact. Then (again by metrizability) we know that E is τ-compact, and hence by Lemma 2.1 it is weakly precompact. Therefore, by KREIN'S THEOREM (as it appears in Bourbaki, see [1, §IV.5]), the closed **balanced** convex hull of E is weakly compact. Since this hull is $\overline{T(\mathsf{B})}$, T is weakly compact. $\square$

The following notation will be used frequently. Given a subset $S \subseteq \mathbb{Z}_+$ and $n \in \mathbb{Z}_+$, we denote by $S - n$ the set $\{t \in \mathbb{Z}_+ : t + n \in S\}$.

We need the following definition, due to Ruppert [11] in a more general setting.

Definition 2.3 ([11]) Let $S \subseteq \mathbb{Z}_+$. We say that S is translation-finite (TF for short) if, for every sequence $n_1 < n_2 < \dots$ in $\mathbb{Z}_+$, there exists k such that

$$\bigcap_{i=1}^{k} (S - n_i) \quad \text{is finite or empty.} \tag{2.1}$$

(In the later paper [3], TF-sets are called "R_W sets"; we believe that for our purposes the older terminology of Ruppert is more suggestive and apposite.)

TF-sets were introduced by Ruppert in the investigation of weakly almost periodic subsets of semigroups. Recall that a bounded function f on a semigroup $\mathbb{S}$ is said to be weakly almost periodic if the set of translates $\{s \cdot f : s \in \mathbb{S}\} \cup \{f \cdot s : s \in \mathbb{S}\}$ is weakly precompact in $\ell^\infty(\mathbb{S})$. Specializing to the case where the semigroup in question is $\mathbb{Z}_+$, one of Ruppert's results can be stated as follows.

Theorem 2.4 ([11]) *Let $S \subseteq \mathbb{Z}_+$. Then S is TF if and only if all bounded functions $S \to \mathbb{C}$ are WAP as elements of $\ell^\infty(\mathbb{Z}_+)$. In particular, if S is a TF-set then the indicator function of S belongs to $\mathsf{WAP}(\mathbb{Z}_+)$.*

It is sometimes convenient to use an alternative phrasing of the original definition (see [3] for instance).

Lemma 2.5 *Let $S \subseteq \mathbb{Z}_+$. Then S is non-TF if and only if there are strictly increasing sequences $(a_n)_{n\geq 1}, (b_n)_{n\geq 1} \subset \mathbb{Z}_+$ such that $\{a_1, \dots, a_n\} \subseteq S - b_n$ for all n.*

Proof. Suppose that there exist sequences (a_n), (b_n) as described. Then for every $n \geq m \in \mathbb{N}$ we have $\{a_1, \dots, a_m\} \subseteq S - b_n$. Hence

$$\{b_n \ : \ n \geq m\} \subseteq \bigcap_{k=1}^{m} (S - a_k)$$

where the set on the left hand side is infinite, for all m. Thus S is non-TF.

Conversely, suppose S is non-TF: then there is a sequence $a_1 < a_2 < \dots$ in $\mathbb{Z}_+$ such that $\bigcap_{j=1}^{k}(S - a_j)$ is infinite for all $k \in \mathbb{N}$. Let $b_1 \in S - a_1$. We inductively construct a sequence (b_n) as follows: if we have already chosen b_n, then since $\bigcap_{k=1}^{n+1}(S - a_k)$ is infinite it contains some $b_{n+1} > b_n$. By construction the sequences (a_n) and (b_n) are strictly increasing, and $\{a_1, \dots, a_n\} \subseteq S - b_n$ for all n. $\square$

Our main result, which characterizes weak compactness of D_ψ in terms of ψ, is as follows.

Theorem 2.6 *Let $\psi \in \ell^\infty(\mathbb{N})$. Then D_ψ is weakly compact if and only if, for all $\varepsilon > 0$, the set $S_\varepsilon := \{n \in \mathbb{N} : |\psi_n| > \varepsilon\}$ is TF.*

It is not clear to the authors if one can deduce Theorem 2.6 in a "soft" way from Ruppert's characterization (Theorem 2.4). Instead, we give a direct argument. The proof naturally breaks into two parts, both of which can be carried out in some generality.

Given $\psi \in \ell^\infty(\mathbb{Z}_+)$ and $M \in \ell^\infty(\mathbb{Z}_+ \times \mathbb{Z}_+)$, define $T_\psi^M : \ell^1(\mathbb{Z}_+) \to \ell^\infty(\mathbb{Z}_+)$ by

$$T_\psi^M(\delta_j)(\delta_k) = M_{jk}\psi_{j+k} \qquad (j, k \in \mathbb{Z}_+). \tag{2.2}$$

In particular, if we take $\psi \in \ell^\infty(\mathbb{N})$, identified with $(0, \psi_1, \psi_2, \dots) \in \ell^\infty(\mathbb{Z}_+)$, and take M_{jk} to be 0 for $j = k = 0$ and to be $j/(j+k)$ otherwise, then $T_\psi^M \equiv D_\psi$.

Proposition 2.7 *Let $\psi \in \ell^\infty(\mathbb{Z}_+)$ be such that S_ε is TF for all $\varepsilon > 0$. If $M \in \ell^\infty(\mathbb{Z}_+ \times \mathbb{Z}_+)$ is such that $\lim_{k\to\infty} M_{jk} = 0$ for all j, then T_ψ^M is weakly compact.*

Proof. To ease notation we write T_ψ for T_ψ^M throughout this proof. Note that the condition on M implies that T_ψ takes values in $c_0(\mathbb{Z}_+)$.

Define $\psi_\varepsilon \in \ell^\infty(\mathbb{Z}_+)$ as follows: set $(\psi_\varepsilon)_n$ to be ψ_n if $n \in S_\varepsilon$ and 0 otherwise. Then ψ_ε is supported on S_ε and $T_{\psi_\varepsilon} \to T_\psi$ as $\varepsilon \to 0$. It therefore suffices to prove that if ψ has TF support, then T_ψ is weakly compact.

Let $\psi \in \ell^\infty$ be supported on a TF set S. Let $(j_n)_{n\geq 1} \subset \mathbb{Z}_+$ be a strictly increasing sequence and set $(j_{0,n}) = (j_n)$, $k_0 = 0$. For each $i \geq 1$ we specify an integer $k_i \in \mathbb{Z}_+$ and a sequence $(j_{i,n})_{n\geq 1} \subset \mathbb{Z}_+$ recursively, as follows. If there exists $k \in \mathbb{Z}_+ \setminus \{k_0, \dots, k_{i-1}\}$ such that $j_{i-1,n} + k \in S$ for infinitely many $n \in \mathbb{N}$, let k_i be some such k. Otherwise, let $k_i = k_{i-1}$. Let $(j_{i,n})_{n\geq 1}$ be the enumeration of the set

$$\{j_{i-1,n} : n \in \mathbb{N}, j_{i-1,n} + k_i \in S\}$$

with $j_{i,n} < j_{i,n+1}$ for each $n \in \mathbb{N}$. Then, by induction on i, $(j_{i,n})_n$ is a subsequence of $(j_n)_n$ and, for each $l \in \{1, \dots, i\}$, and each $n \in \mathbb{N}$, we have $j_{i,n} + k_l \in S$.

In particular, for each $i \in \mathbb{N}$, $\{j_{i,i}+k_1, \ldots, j_{i,i}+k_i\} \subset S$. Hence, by our assumption that S is TF, the set $\{k_i : i \in \mathbb{Z}_+\}$ is finite. Let i_0 be the smallest i for which $k_i = k_{i+1}$: then for each $k \in \mathbb{Z}_+ \setminus \{k_0, \ldots, k_{i_0}\}$, there are only finitely many $n \in \mathbb{N}$ such that $j_{i_0,n}+k \in S$.

Let $K = \{k_0, \ldots, k_{i_0}\}$. By the HEINE-BOREL THEOREM, there exists a subsequence $(j_{n(m)})_m$ of $(j_{i_0,n})_n$ such that, for each $k \in K$, $\lim_{m\to\infty} T_\psi(\delta_{j_{n(m)}})(\delta_k)$ exists. Moreover, by the previous paragraph, for each $k \in \mathbb{Z}_+ \setminus K$ there exist at most finitely many m such that $j_{n(m)}+k \in S$; hence there exists $m(k)$ such that

$$T_\psi(\delta_{j_{n(m)}})(\delta_k) = M_{j_{n(m)},k}\psi(j_{n(m)}+k) = 0 \qquad \text{for all } m \geq m(k).$$

Thus $T_\psi(\delta_{j_{n(m)}})$ converges pointwise to some function supported on K, and the result follows by Lemma 2.2. $\square$

Proposition 2.8 *Let $M \in \ell^\infty(\mathbb{Z}_+ \times \mathbb{Z}_+)$ satisfy*

$$\lim_{k\to\infty} M_{jk} = 0 \textit{ for all } j, \textit{ and } \inf_k \liminf_{j\to\infty} |M_{jk}| = \eta > 0\,. \tag{2.3}$$

Let $\psi \in \ell^\infty$, and suppose that T_ψ^M is weakly compact. Then S_ε is TF for all $\varepsilon > 0$.

Proof. We first note that (2.3) implies that T_ψ^M has range contained in $c_0(\mathbb{Z}_+)$. Suppose the result is false: then there exists $\varepsilon > 0$ such that S_ε is non-TF. Hence, by Lemma 2.5, there exist strictly increasing sequences $(a_n), (b_n) \subset \mathbb{Z}_+$ such that

$$\{a_1+b_n, \ldots, a_n+b_n\} \subset S_\varepsilon \qquad \text{for all } n.$$

Now

$$|T_\psi^M(\delta_{b_n})(\delta_{a_m})| = |M_{b_n,a_m}||\psi_{a_m+b_n}| \geq |M_{b_n,a_m}|\varepsilon \qquad \text{for all } m \leq n;$$

so, by the hypothesis (2.3), we have

$$\inf_m \liminf_n |T_\psi^M(\delta_{b_n})(\delta_{a_m})| \geq \eta\varepsilon\,. \tag{2.4}$$

Since T_ψ^M is weakly compact, by Lemma 2.2 the sequence $(T_\psi(\delta_{b_n}))$ has a subsequence that converges pointwise to some $\phi \in c_0(\mathbb{Z}_+)$. But by (2.4) we must have $\inf_m |\phi_{a_m}| \geq \eta\varepsilon$, so that $\phi \notin c_0(\mathbb{Z}_+)$. Hence we have a contradiction and the proof is complete. $\square$

Proof of Theorem 2.6. Sufficiency of the stated condition follows from Proposition 2.7; necessity, from Proposition 2.8, once we observe that $\lim_k j/(j+k) = 0$ for all j, and $\lim_j j/(j+k) = 1$ for all k. $\square$

Remark 2.9 The set $S = \{2^n \; : \; n \in \mathbb{N}\}$ is TF. In fact, it is not hard to show it has the following stronger property:

$$\text{for } \textit{every } n \in \mathbb{N}, \text{ the set } S \cap (S - n) \text{ is finite or empty.} \qquad (\dagger)$$

We therefore obtain another proof that the derivation constructed in Proposition 1.1 is weakly compact.

Subsets of $\mathbb{Z}_+$ satisfying the condition $(\dagger)$ seem not to have an agreed name. They were called T-sets in work of Ramirez [10], and for ease of reference we shall use his terminology.

3 Comparing the TF-property with other notions of size

Let $S \subset \mathbb{Z}_+$. For $n \in \mathbb{N}$ we define $f_S(n)$ to be the nth member of S.

Proposition 3.1 *Let $S \subset \mathbb{Z}_+$. Then there exists a non-TF $R \subset \mathbb{Z}_+$ with $f_R(n) > f_S(n)$ for all n.*

Proof. For $n \in \mathbb{Z}_+$, let $t_n = \frac{1}{2}n(n+1)$ be the nth triangular number, so that $t_n = t_{n-1} + n$ for all $n \in \mathbb{N}$. Each $n \in \mathbb{N}$ has a unique representation as $n = t_{k-1} + j$ where $k \geq 1$ and $1 \leq j \leq k$. Enumerate the elements of S in increasing order as $s_1 < s_2 < s_3 < \ldots$, and define a sequence $(r_n)_{n \geq 1}$ by

$$r_{t_{k-1}+j} = s_{t_k} + j \qquad (1 \leq j \leq k)$$

as indicated by the following diagram:

$$\begin{array}{lll} r_1 = s_1 + 1 & & \\ r_2 = s_3 + 1 & r_3 = s_3 + 2 & \\ r_4 = s_6 + 1 & r_5 = s_6 + 2 & r_6 = s_6 + 3 \\ \quad\vdots & & \end{array}$$

Since the sequence (s_n) is strictly increasing,

$$s_{t_k} \geq s_{t_{k-1}} + (t_k - t_{k-1}) = s_{t_{k-1}} + k \qquad (k \in \mathbb{N}),$$

and so the sequence (r_n) is strictly increasing. Put $R = \{r_n \; : \; n \in \mathbb{N}\}$: then clearly $f_R(n) = r_n > s_n = f_S(n)$ for all n. Finally, since $\bigcap_{j=1}^m (R - j) \supseteq \{s_{t(k)} \; : \; k \geq m\}$ for all m, R is not a TF-set. ☐

On the other hand, we can find T-sets S such that f_S grows at a "nearly linear" rate.

Proposition 3.2 *Let $g : \mathbb{N} \to \mathbb{Z}_+$ be any function such that $g(n)/n \to \infty$. Then there is a strictly increasing sequence $a_1 < a_2 < \ldots$ in $\mathbb{Z}_+$, such that $\{a_n : n \in \mathbb{N}\}$ is a T-set, while $a_n < g(n)$ for all but finitely many n.*

Proof. Let $N \in \mathbb{N}$ and set k_N to be the smallest natural number such that $g(n) > Nn$ for all $n > k_N$. We now construct our sequence (a_n) recursively. Set $a_0 = 0$ and suppose that $a_0, \ldots, a_n$ have been defined: if $a_n < k_1$ set $a_{n+1} = a_n + 1$; otherwise, if $k_N \leq a_n < k_{N+1}$ for some $N \in \mathbb{N}$, set $a_{n+1} = a_n + N$. Thus, the elements of $[k_N, k_{N+1}] \cap \{a_n : n \in \mathbb{N}\}$ are in arithmetic progression with common difference N.

A simple induction gives that if $k_N \leq n < k_{N+1}$, then $a_n \leq Nn$. Since for $n \geq k_N$ we also have that $g(n) > Nn$, it follows that, for all $n \geq k_1$, $a_n < g(n)$.

Finally, let $i, j \in \mathbb{N}$. If $a_i - j \in \{a_n : n \in \mathbb{N}\}$ it follows that $a_i < k_{j+1} + j$. Thus, $\{a_n : n \in \mathbb{N}\}$ is a T-set. $\square$

Remark 3.3 Given that infinite arithmetic progressions are the most obvious examples of non-TF sets, it may be worth noting that if $g(n)/n^2 \to 0$, the T-set constructed in the proof of Proposition 3.2 contains arbitrarily long arithmetic progressions.

The previous two results indicate that the growth of a subset in $\mathbb{Z}_+$ tells us nothing, on its own, about whether or not it is TF. The main result of this section shows that, nevertheless, certain kinds of *density property* are enough to force a set to be non-TF. First we need some definitions.

Definition 3.4 Let $S \subset \mathbb{Z}_+$. The upper Banach density[1] of S, denoted by $\mathbf{Bd}(S)$, is

$$\mathbf{Bd}(S) := \lim_{d\to\infty} \max_n d^{-1} |S \cap \{n+1, \ldots, n+d\}|$$

(The limit always exists, by a subadditivity argument.)

For example, the set R constructed in the proof of Proposition 3.1 satisfies $R \supseteq \{s_{t(m)} + 1, \ldots, s_{t(m)} + m\}$ for all m, and so has a Banach density of 1.

Proposition 3.5 *Let $S \subset \mathbb{Z}^+$ and suppose $\mathbf{Bd}(S) > 0$. Then S is not TF.*

The converse clearly fails: for example, the set $S = \{2^i + j^2 : i \in \mathbb{N}, j \in \{0, \ldots, i\}\}$ is not TF, but has Banach density zero.

The proof of Proposition 3.5 builds on some preliminary lemmas, which in turn require some notation. Fix once and for all a set $S \subset \mathbb{Z}_+$ with strictly positive Banach density, and choose $\varepsilon \in (0, 1)$ such that $\mathbf{Bd}(S) > \varepsilon$.

For shorthand, we say that a subset $X \subseteq \mathbb{Z}_+$ is recurrent in S if there are infinitely many $n \in \mathbb{N}$ such that $X + n \subset S$. For each $d \in \mathbb{N}$, let

$$\mathcal{V}_d = \{X \subset \mathbb{N} \ : \ X \text{ is recurrent in } S \text{ and } d \geq |X| \geq d\varepsilon\}.$$

Lemma 3.6 *For every $d \in \mathbb{N}$, $\mathcal{V}_d$ is non-empty.*

[1] What we call "Banach density" is also referred to as upper Banach density, and is in older sources given a slightly different but equivalent definition. Some background and remarks on the literature can be found in [6, §1], for example.

Proof. The key step is to prove that the set $\{i \in \mathbb{N} : |S \cap \{i+1, \ldots, i+d\}| \geq d\varepsilon\}$ is infinite, which we do by contradiction. For, suppose it is finite, with cardinality N, say: then for any $j, n \in \mathbb{N}$ we have

$$\begin{aligned}(jd)^{-1}|S \cap \{n+1, \ldots, n+(jd)\}| &= j^{-1} \sum_{m=0}^{j-1} d^{-1}|S \cap \{n+md+1, \ldots, n+(m+1)d\}| \\ &\leq j^{-1}(N + (j-N)\varepsilon)\,,\end{aligned}$$

so that

$$\mathbf{Bd}(S) = \lim_j (jd)^{-1} \sup_n |S \cap \{n+1, \ldots, n+(jd)\}| \leq \limsup_j j^{-1}(N+(j-N)\varepsilon) = \varepsilon\,,$$

which contradicts our original choice of ε.

It follows that there exists a strictly increasing sequence $i_1 < i_2 < \ldots$ in $\mathbb{N}$, such that $|S \cap \{i_n + 1, \ldots, i_n + d\}| \geq d\varepsilon$ for all n. Since there are at most finitely many subsets of $\{1, \ldots, d\}$, by passing to a subsequence we may assume that the sequence of sets $((S - i_n) \cap \{1, \ldots, d\})_{n \geq 1}$ is constant, with value X say. Clearly $X \in \mathcal{V}_d$, which completes the proof. $\square$

Lemma 3.7 *There exists a sequence $1 = d_1 < d_2 < \ldots$ in $\mathbb{N}$ such that, for every $j \in \mathbb{N}$ and any $X \in \mathcal{V}_{d_{j+1}}$, there exists $Y \in \mathcal{V}_{d_j}$ such that $Y \subseteq X$ and $\max(Y) < \max(X)$.*

Proof. Put $d_1 = 1$ and choose $N_1 \in \mathbb{N}$ such that $N_1 > \varepsilon^{-1}$. We then inductively construct our sequence (d_n) as follows: if we have already defined d_j for some $j \in \mathbb{N}$, let a_j be the largest non-negative integer such that $a_j < d_j\varepsilon$. Then choose $N_j \in \mathbb{N}$, $N_j \geq 2$, large enough that

$$\frac{1}{N_j}\left[(N_j - 1)\frac{a_j}{d_j} + 1\right] < \varepsilon\,, \tag{3.1}$$

and set $d_{j+1} = N_j d_j$.

To show that this sequence has the required properties, let $j \in \mathbb{N}$. Given $X \in \mathcal{V}_{d_{j+1}}$, put $x_0 = \min(X)$, and for $m = 0, 1, \ldots, N_j - 1$ put

$$Y_m = X \cap \{x_0 + md_j, \ldots, x_0 + (m+1)d_j - 1\}.$$

Since $|X| \leq d_{j+1}$ and $\min(X) = x_0$, the sets $Y_0, \ldots, Y_{N_j - 1}$ form a partition of X. Since X is recurrent in S, so is each of the subsets Y_m, and by construction $|Y_m| \leq d_j$ for all m.

We claim that there exist $m(1) < m(2) \in \{0, 1, \ldots, N_j - 1\}$ such that $Y_{m(1)}$ and $Y_{m(2)}$ have cardinality $\geq d_j\varepsilon$. If this is the case then $Y_{m(1)} \in \mathcal{V}_{d_j}$ and $\max(Y_{m(1)}) < \min(Y_{m(2)}) \leq \max(X)$, so that we may take $Y = Y_{m(1)}$ in the statement of the lemma.

Suppose the claim is false. Then at least $N_j - 1$ of the sets $Y_0, \ldots, Y_{N_j - 1}$ have cardinality $< d_j\varepsilon$, and hence (by the definition of a_j) at least $N_j - 1$ of these sets have

cardinality $\leq a_j$. Now since $X \subseteq \{x_0, \ldots, x_0 + d_{j+1} - 1\}$, we have $X = Y_0 \sqcup \ldots \sqcup Y_{N_j - 1}$, and so

$$N_j d_j \varepsilon = d_{j+1}\varepsilon \leq |X| = \sum_{m=0}^{N_j-1} |Y_m| \leq d_j + (N_j - 1)a_j .$$

On dividing through by $N_j d_j$, we obtain a contradiction with (3.1), and our claim is proved. $\square$

The final ingredient in our proof of Proposition 3.5 is purely combinatorial: it is a version of 'KŐNIG'S INFINITY LEMMA', which we isolate and state for convenience. We shall paraphrase the formulation given in [5, Lemma 8.1.2], and refer the reader to that text for the proof.

Lemma 3.8 *Let $\mathcal{G}$ be a graph on a countably infinite vertex set V, and let $V = \coprod_{j\geq 1} V_j$ be a partition of V into mutually disjoint, non-empty finite subsets. Suppose that for each $j \geq 1$, every $v \in V_{j+1}$ has a neighbour in V_j. Then there exists a sequence $(v_n)_{n\geq 1}$, with $v_n \in V_n$ for each n, such that v_{n+1} is a neighbour of v_n.*

Proof of Proposition 3.5. Let (d_j) be the sequence from Lemma 3.7. For each j, let V_j be the set of all subsets of $\{1, \ldots, d_j\}$ which are also members of $\mathcal{V}_{d_j}$. The proof of Lemma 3.6 shows that V_j is non-empty, and clearly it is a finite set.

Regard $\coprod_{j\geq 1} V_j$ as the vertex set for a graph, whose edges are defined by the following rule: for each $j \in \mathbb{N}$ and $Y \in V_j$, $X \in V_{j+1}$, join X to Y with an edge if and only if there exists $m \in \mathbb{Z}_+$ with $Y + m \subseteq X$ and $\max(Y) + m < \max(X)$. Then by Lemma 3.7, every element of V_{j+1} has a neighbour in V_j. Hence, by Lemma 3.8, there exists a sequence $(Y_j)_{j\geq 1}$ of finite subsets of $\mathbb{N}$, and a sequence $(m_j) \subset \mathbb{Z}_+$, such that

(i) $Y_j \in V_j$ for all j;

(ii) $Y_j + m_j \subseteq Y_{j+1}$ and $\max(Y_j) + m_j < \max(Y_{j+1})$ for all j.

Now put $X_1 = Y_1$ and put $X_{j+1} = Y_{j+1} - (m_j + \cdots + m_1) \subset \mathbb{N}$ for $j \geq 1$. An easy induction using *both parts* of (ii) above shows that $X_j \subsetneq X_{j+1}$ for all j. Since each Y_i is recurrent in S, so is each X_i, and hence there exist infinitely many n such that $X_i + n \subset S$. We may therefore inductively construct $n_1 < n_2 < \ldots$ such that $X_i + n_i \subset S$ for all i.

Pick $c_1 \in X_1$ and for each i pick $c_{i+1} \in X_{i+1} \setminus X_i$; then for all $1 \leq i \leq j$ we have $c_i + n_j \in X_j + n_j \subset S$; and since the set $\{c_i : i \in \mathbb{N}\}$ is infinite, by Lemma 2.5 S is not TF. $\square$

4 Closing thoughts

We finish with some remarks and questions that this work raises. Here and in the appendix, it will be convenient to abuse notation as follows: given $S \subseteq \mathbb{N}$, we write D_S for the derivation D_{χ_S}, where χ_S is the indicator function of S. For example, with this notation $D_{\mathbb{N}} \equiv D_{\mathbf{1}}$.

Combinatorics of TF subsets of $\mathbb{Z}_+$

We have been unable to find much in the literature on the combinatorial properties of TF subsets of $\mathbb{Z}_+$. Here are some elementary facts.

- Let $k \in \mathbb{N}$; then S is TF if and only if $S + k$ is.
- Finite unions of T-sets are TF.
- The set of odd numbers is non-TF, as is the set of even numbers. In particular, the complement of a non-TF set can be non-TF.
- Subsets of TF sets are TF. (Immediate from the definition.) In particular, the intersection of two TF sets is TF.
- If S and T are TF then so is $S \cup T$.

The last of these observations follows immediately if we grant ourselves Ruppert's result (Theorem 2.4 above). It also follows from our Theorem 2.6: for if S and T are TF subsets of $\mathbb{Z}_+$, then since $S+1$, $(S\cap T)+1$ and $T+1$ are also TF, the derivations D_{S+1}, D_{T+1} and $D_{(S\cap T)+1}$ are all weakly compact; whence

$$D_{(S\cup T)+1} = D_{S+1} - D_{(S\cap T)+1} + D_{T+1}$$

is also weakly compact, so that by the other direction of Theorem 2.6, $(S\cup T)+1$ and hence $S \cup T$ are TF. It also seems worth giving a direct, combinatorial proof, which to our knowledge is not spelled out in the existing literature (cf. [11, Remark 18]).

Proof. Let A_1, A_2 be subsets of $\mathbb{Z}_+$ and suppose that $A_1 \cup A_2$ is not TF. By Lemma 2.5 there exist $a_1 < a_2 < \dots$ and $b_1 < b_2 < \dots$ in $\mathbb{Z}_+$, such that $\{a_m + b_n \ : \ 1 \leq m \leq n\} \subseteq A_1 \cup A_2$. Let

$$\begin{aligned} E &= \{(m,n) \in \mathbb{N}^2 \ : \ m < n, a_m + b_n \in A_1\}\,, \\ F &= \{(m,n) \in \mathbb{N}^2 \ : \ m < n, a_m + b_n \in A_2 \setminus A_1\}\,. \end{aligned}$$

The sets E and F can be regarded as a partition of the set of 2-element subsets of $\mathbb{N}$. Hence, by Ramsey's theorem [5, Theorem 9.1.2], there exists either an infinite set $S \subset \mathbb{N}$ such that $\{(x,y) \in S^2 \ : \ x < y\} \subseteq E$, or an infinite set $T \subset \mathbb{N}$ such that $\{(x,y) \in T^2 \ : \ x < y\} \subseteq F$.

In the former case, enumerate S as $s_0 < s_1 < s_2 < \dots$, and put $c_j = a_{s_{j-1}}$, $d_j = b_{s_j}$ for $j \in \mathbb{N}$. Then $c_i + d_j \in A_1$ for all $1 \leq i \leq j$; therefore, by Lemma 2.5, A_1 is not TF.

In the latter case, a similar argument shows that A_2 is not TF. We conclude that at least one of A_1 and A_2 is non-TF, which proves the desired result. ☐

Generalizations to other (semigroup) algebras?

We have relied heavily on the convenient parametrization of $\mathrm{Der}(\ell^1(\mathbb{Z}_+), \ell^1(\mathbb{Z}_+)^*)$ by elements of $\ell^\infty(\mathbb{N})$. There are analogous parametrizations for $\mathbb{Z}_+^k$, where $k \geq 2$, but it is not clear to the authors if they allow one to obtain reasonable higher-rank analogues of Theorem 2.6.

We can at least make one general observation.

Definition 4.1 Let A be a Banach algebra and X a Banach A-bimodule. If $x \in X$, we say that x is a weakly almost periodic element of X if both $a \mapsto ax$ and $a \mapsto xa$ are weakly compact as maps from A to X. The set of all weakly almost periodic elements of X will be denoted by $\mathsf{WAP}(X)$.

Combining Proposition 2.8 with Ruppert's result (Theorem 2.4), we see that if D_ψ is weakly compact then $\psi \in \mathsf{WAP}(\ell^\infty(\mathbb{Z}_+))$, where we identify $\psi \in \ell^\infty(\mathbb{N})$ with $(0, \psi_1, \psi_2, \dots) \in \ell^\infty(\mathbb{Z}_+)$. This is a special case of a more general result.

Proposition 4.2 *Let A be a unital Banach algebra, let $D : A \to A^*$ be a weakly compact derivation, and let $\psi \in A^*$ be the functional $D(\cdot)(1)$. Then $\psi \in \mathsf{WAP}(A^*)$.*

Proof. Let $\kappa : A \to A^{**}$ be the canonical embedding. By GANTMACHER'S THEOREM, $D^* : A^{**} \to A^*$ is weakly compact, so $D^*\kappa$ is also weakly compact. Note that $D^*\kappa(a) = D(\cdot)(a)$ for all $a \in A$.

Let $a \in A$, and consider $\psi \cdot a \in A^*$. For each $b \in A$ we have

$$(\psi \cdot a)(b) = \psi(ab) = D(ab)(1) = D(a)(b) + D(b)(a) \;;$$

thus $\psi \cdot a = D(a) + D^*\kappa(a)$. Since D and $D^*\kappa$ are weakly compact, this shows that the map $a \mapsto \psi \cdot a$ is weakly compact. A similar argument shows that the map $a \mapsto a \cdot \psi$ is weakly compact, and so $\psi \in \mathsf{WAP}(A^*)$ as claimed. $\square$

When $A = \ell^1(\mathbb{S})$ is the convolution algebra of a discrete monoid $\mathbb{S}$, we may regard $A^* = \ell^\infty(\mathbb{S})$ as an algebra with respect to pointwise multiplication. The previous proposition shows that the functional $D(\cdot)(\delta_e)$ lies in $\mathsf{WAP}(A^*)$: is it the case that $hD(\cdot)(\delta_e)$ lies in $\mathsf{WAP}(A^*)$ for every $h \in \ell^\infty(\mathbb{S})$?

Acknowledgments

The authors thank the referee and N. J. Laustsen (as editorial adviser) for a close reading of the text and for useful corrections.

A A weakly compact derivation which is not p-summing

Definition A.1 Let X and Y be Banach spaces and let $p \in [1, \infty)$. We say that a bounded linear map $T : X \to Y$ is p-summing if there exists $C > 0$ such that:

$$\sum_{j=1}^m \|Tx_j\|^p \leq C^p \sup_{\phi \in X^*, \|\phi\| \leq 1} \sum_{j=1}^m |\phi(x_j)|^p, \qquad \text{for all } m \in \mathbb{N} \text{ and } x_1, \dots, x_m \in X. \tag{A.1}$$

The least such C is denoted by $\pi_p(T)$. If no such C exists (i.e. if T is not p-summing) we write $\pi_p(T) = \infty$.

Recall that in Proposition 1.1, taking S to be the set of integer powers of 2 gives a derivation D_S that is 1-summing. In this appendix, we construct a T-set A such that D_A, while weakly compact, is not p-summing for any finite p. To do this, we shall need

some standard general results, which are collected in the following proposition for ease of reference.

Proposition A.2 *Let X and Y be Banach spaces and let $T \in B(X,Y)$.*

(i) *Let $1 \leq p \leq q < \infty$. Then $\pi_p(T) \geq \pi_q(T)$.*

(ii) *If T is p-summing for some $p \in [1,\infty)$, then it is weakly compact.*

We refer to [4] for the proofs: part (i) may be found as [4, Theorem 2.8]; and part (ii) follows from the Pietsch factorization theorem, see [4, Theorem 2.17].

We now specialize to operators of the form D_ψ. The key observation is the following.

Lemma A.3 *Let $\psi \in \ell^\infty(\mathbb{N})$ and $p \in [1,\infty)$ and $K < \pi_p(D_\psi)$. There exists $N \in \mathbb{N}$ depending on ψ, on p and on K, such that for each $\psi' \in \ell^\infty(\mathbb{N})$ satisfying $\psi(k) = \psi'(k)$ for all $k < N$, we have $\pi_p(D_{\psi'}) > K$.*

Proof. There are $x_1, \ldots, x_m \in \ell^1(\mathbb{Z}_+)$ such that

$$\sum_{j=1}^{m} \|D_\psi(x_j)\|^p > K^p \sup_{\phi \in \ell^\infty, \|\phi\| \leq 1} \sum_{j=1}^{m} |\phi(x_j)|^p. \tag{A.2}$$

Without any loss of generality we may take $x_1, \ldots, x_n \in c_{00}$; write $x_j = \sum_{i=0}^{l(j)} \alpha_{i,j}\delta_i$. For each $j \in \{1, \ldots, m\}$, since $D_\psi(x_j) \in c_0$, there exists $n(j) \in \mathbb{N}$ with $|D_\psi(x_j)(\delta_{n(j)})| = \|D_\psi(x_i)\|$.

Fix $N > \max\{l(1)+n(1), \ldots, l(m)+n(m)\}$, and let $\psi' \in \ell^\infty(\mathbb{N})$ be such that $\psi(k) = \psi'(k)$ for all $k < N$. Observe now that for each j we have

$$\begin{aligned} D_\psi(x_j)(\delta_{n(j)}) &= \sum_{i=1}^{l(j)} \alpha_{i,j} \frac{i}{i+n(j)} \psi(i+n(j)) \\ &= \sum_{i=1}^{l(j)} \alpha_{i,j} \frac{i}{i+n(j)} \psi'(i+n(j)) \quad = D_{\psi'}(x_j)(\delta_{n(j)}) \,. \end{aligned}$$

Therefore,

$$\sum_{j=1}^{m} \|D_{\psi'}(x_j)\|^p \geq \sum_{j=1}^{m} |D_{\psi'}(x_j)(\delta_{n(j)})|^p = \sum_{j=1}^{m} |D_\psi(x_j)(\delta_{n(j)})|^p = \sum_{j=1}^{m} \|D_\psi(x_j)\|^p.$$

Combining this with Equation (A.2) yields $\pi_p(D_{\psi'}) > K$, and the result follows. □

We can now give the promised example.

Theorem A.4 *There exists a T-set A such that D_A is not p-summing for any $p < \infty$.*

Proof. The set A will be the disjoint union of a sequence of finite arithmetic progressions whose "skip size" tends to infinity. For each $k \in \mathbb{Z}_+$, we shall construct, recursively, $A(k) \subset \mathbb{N}$ and $c_k \in \mathbb{Z}_+$ such that

(a) $c_k > \max A(k)$;

(b) $\pi_k(D_B) > k$ for each $B \subset \mathbb{N}$ satisfying $B \cap \{1,\ldots,c_k\} = A(k) \cap \{1,\ldots,c_k\}$;

(c) $A(k) \supset A(k-1)$ for all $k \geq 1$.

Let $A(0) = \emptyset$ and $c_0 = 0$. For each $k \in \mathbb{N}$ assume that we have already defined $A(k-1) \subset \mathbb{N}$ and $c_{k-1} \in \mathbb{N}$ satisfying conditions (a) and (b). Let $S := A(k-1) \cup (c_{k-1} + k\mathbb{N})$. Since S contains an infinite arithmetic progression, it is non-TF. Hence by Theorem 2.6 D_S is not weakly compact, and so by part (i) of Proposition A.2 it is not k-summing. In particular, $\pi_k(D_S) > k$, so by applying Lemma A.3 with $\psi = \chi_S$, we see that there exists M such that

$$\pi_k(D_{S\cap\{1,\ldots,m\}}) > k \quad \text{for all } m \geq M. \tag{A.3}$$

Choose n such that $c_{k-1} + kn \geq M$, and take

$$A(k) := S \cap \{1,\ldots,c_{k-1}+kn\} = A(k-1) \cup \{c_{k-1}+k, c_{k-1}+2k, \ldots, c_{k-1}+nk\}\,.$$

By construction this choice satisfies condition (c). Applying Lemma A.3 with $\psi = \chi_{A(k)}$, we can choose c_k satisfying conditions (a) and (b), and so our construction may continue.

Set $A := \bigcup_{k=1}^{\infty} A(k)$. Then for each $k \in \mathbb{N}$, $A \cap \{1,\ldots,c_k\} = A(k) \cap \{1,\ldots,c_k\}$ and so $\pi_k(D_A) > k$. Thus by Proposition A.2(i) $\pi_p(D_A) = \infty$ for all $p \in [1,\infty)$. Finally, if we enumerate the elements of A as an increasing sequence $a_1 < a_2 < \ldots$, then $a_{i+1} - a_i \to \infty$; it follows easily that A is a T-set. $\square$

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